PRINCETON UNIVERSITY PHYSICS COMPETITION

GENERAL INFORMATION
Online Part

Guidelines

Student teams will have a total of **one week** to complete the exam from start to finish. We recommend that teams set aside approximately 20 hours to allow enough time for successful completion. All teams are required to submit their response with a cover page listing the title of their work, the date, and the information provided during registration. Additionally, it should include the signatures of all contestants on that team. Each submitted page should also have on it the team ID number and problem number. All other formatting decisions are delegated to the teams themselves, with no one style favored over another. While points will not be deducted for written work, we suggest that teams use a typesetting language (e.g., LaTeX) or a word-processing program (e.g., Microsoft Word/Pages) for convenience. There will be three types of problems on the online exam, with about twenty questions in total:

- **Introductory:** These questions will introduce students to the research topic. No prior knowledge is assumed.
- **Intermediate:** Questions that will bring students up to date with current physics research that assume knowledge of the introductory material and questions.
- **Research:** As the most difficult questions on the exam, research questions will test students' knowledge and creativity in manipulating and interpreting data from current research.

Collaboration Policy

Students participating in the competition may only correspond with other members of their team. No other correspondence is allowed, including: mentors, teachers, professors, and other students. While teams are allowed to use a plethora of online resources, participating students are barred from posting content or asking questions related to the exam. As repeated below, teams are also welcome to utilize the Piazza page at `http://piazza.com/princeton_university_physics_competition/spring2015/pupc` and ask questions in case something is unclear in the assignment.

Resources

Barring violating the collaboration policy, students have access to the following types of resources:

- **Online**: Teams may use any information they find useful on the Internet. However, under no circumstances may they engage in active interactions such as posting content or asking questions regarding the exam.
- **Piazza page**: Teams are encouraged to create an account in Piazza and register in the class at the following URL:

  `http://piazza.com/princeton_university_physics_competition/spring2015/pupc`

  The access code is: **firstedition**

  This way, you will be able to ask questions if you'd like to clarify something.

- **Published Materials**: Teams may take advantage of any published material (print/online)
- **Computational**: Teams may use any computational resources they might find helpful, such as Wolfram Alpha/Mathematica, Matlab, Excel, or lower level programming languages (C++, Java, Python, etc).

Citations

All student submissions with outside material must include numbered citations. We do not prefer any style of citation in particular. Students may find the following guide useful in learning when to cite sourced material:

`http://www.princeton.edu/pr/pub/integrity/pages/cite/`

Submission

Teams must submit their Online Part solutions by e-mailing **pupc.submit@gmail.com** in accordance with the Test Rules before 2 PM Eastern Time (UTC-5) on Saturday, November 22, 2014. Teams will not be able to submit their solutions to the Online Part at any later time. Regardless of internal formatting, solutions should be submitted as a single PDF document with the ".pdf" extension. The e-mail must contain your team ID in the Subject field. Only one person per team, identified as the "team manager" during registration, should send this e-mail. (Team managers will receive their team IDs via e-mail after the Online Part is released, by Sunday, November 16.) Each submitted page should also have on it the team ID number and problem number, and the front page should include the following handwritten pledge ("I pledge my Honor that this assignment represents my own work in accordance with regulations set forth herein") followed by the signatures of all contestants in that team. Any discrepancies will be dealt with by the current Director of PUPC.

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Jane Street

USG Undergraduate Student Government

PUPC 2014: Cosmology Online Part

If you choose to submit your solutions for the cosmology problem, you are NOT required to submit the condensed matter problem!

This part will contain background material in several important topics in cosmology, including some that are at the forefront of groundbreaking research in present day.

Some sections offer background material and are followed by several conceptual and/or applied questions that you are expected to answer in any order you choose (but the material is structured in a way that would lead you through the assignment sequence logically).

Good luck!

Cosmology is the scientific study of the large scale properties of the universe as a whole. It aims to understand the origin, evolution and ultimate fate of the universe. The prevailing theory about the origin and evolution of our universe is the so-called Big Bang theory. In this problem, you will explore the main concepts in cosmology.

1. The Scale Of the Universe

Cosmological observations indicate that the universe is homogeneous and isotropic on large scales. Homogeneity is the statement that the universe looks the same at each point, while isotropy states that the universe looks the same in all directions. This is referred to as the Cosmological Principle and we will assume it to be true for the following problem.

A key piece of observational evidence in cosmology is that almost everything in the Universe appears to be moving away from us, and the further away something is, the more rapid its recession appears to be. Yet the universe looks just the same from any point in space. A common analogy is to imagine baking a cake with raisins in it. As the cake rises, the raisins move apart. But from each raisin's perspective, it seems that all the others are receding, and the further away they are the faster that recession is. Because everything is flying away from everything else, we conclude that in the distant past everything in the Universe was much closer together. For all the following questions, we will consider the widely accepted theory that the universe started with an explosion, called the Big Bang.

Hence, the expansion of the universe as seen from any point in the space, can be described by a scalar factor, a(t), which is independent of position. Given a point in space with vector position r_0, its position at any time t will be characterized by the equation:

$$r(t) = a(t)r_0 \tag{1}$$

At the time of Big Bang, $a(0) = 0$, while at the present time, $a(t_0) = 1$.

1.1. Finding Hubble's Law

Hubble's law expresses the relationship between the vector position r and velocity v of any point in space, for which the expansion of the universe is homogeneous and isotropic. The dependence of r on v is expressed through a multiplicative factor called the Hubble constant. Derive the Hubble law, $v = Hr$. Is the Hubble constant the same at all times or does it change with time? Is the Hubble law true for an observer at any point in space?

1.2. The Naive Age of the Universe

The current value of the Hubble constant, H_0, can be experimentally determined by measuring the distance to a galaxy and the velocity with which the galaxy is receding from us. Hence, we can make a gross approximation for the age of the universe, t_0. Create a simple model and express t_0, also referred to as the Hubble time, as a function of H_0. Clearly state your assumptions.

1.3. Redshifting

(a) However, us pesky astronomers like using quantities that do not make nearly as much intuitive sense. Instead of using the scale factor, astronomers love using another quantity called redshift. Although not as intuitive, the quantity redshift z can be easily extracted from observational data. Look up and explain the spectroscopic definition of redshift $z$and its relationship with the receding velocity v of the observed objects. Plotted below in Figure 1 is the spectrum of a distant galaxy with a powerful supermassive black hole at its center. Try to understand the spectrum with the information given in the caption, and calculate the redshift of this galaxy. Knowing the redshift, can you also name some of the other emission lines on the spectrum?

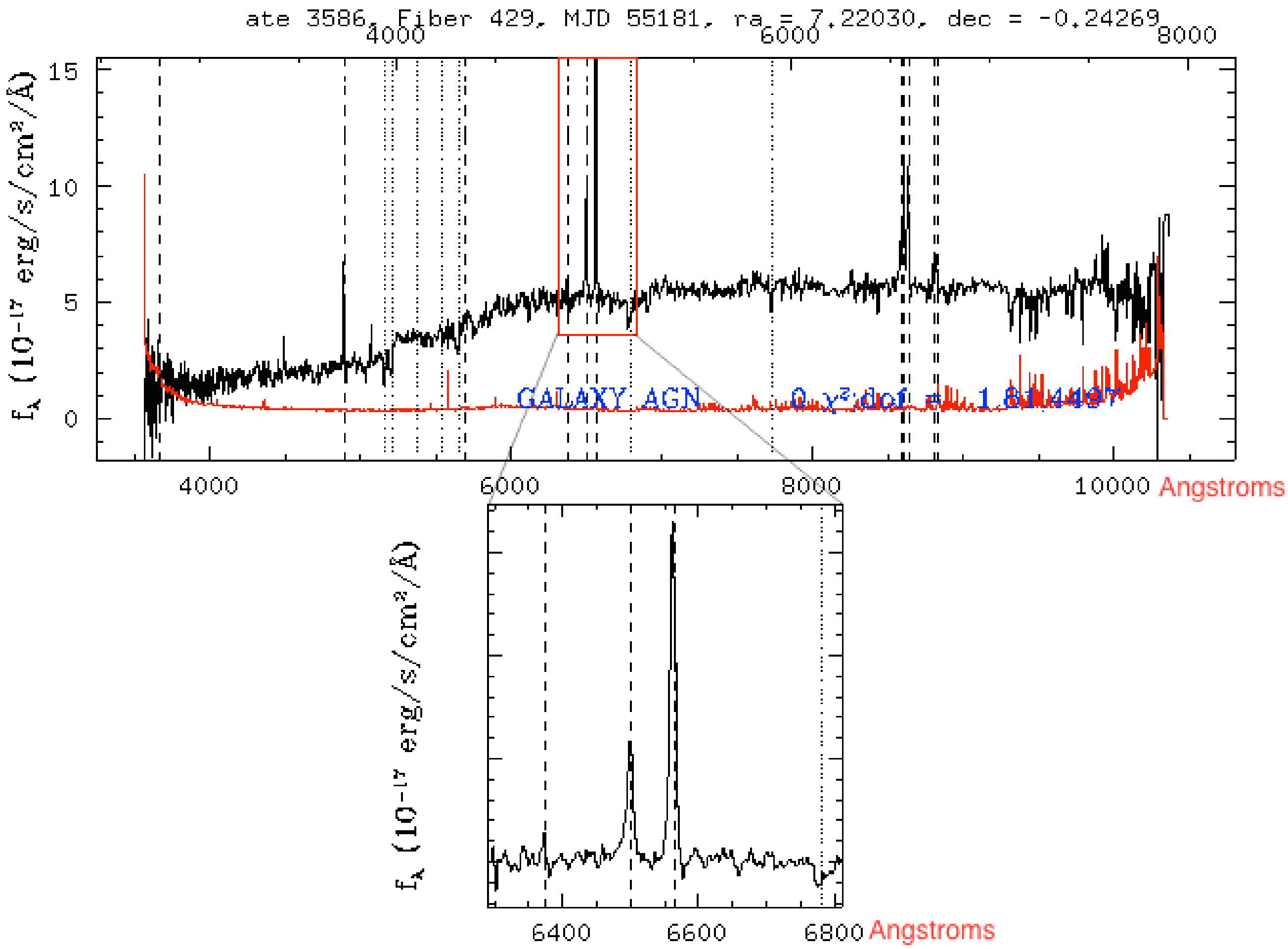

Figure 1: The upper panel is the spectrum of this galaxy given as the energy flux received by the telescope per wavelength (in Angstroms $\mathring{A}$). The red curve towards the bottom denote the data uncertainty. The dashed lines mark out the significant features that astronomers are interested in. The lower panel is a zoom in on the part of spectrum selected by the red box. Now an experienced astronomer would immediately tell you that the two strongest emission lines in the lower panels are the [OIII] $\lambda = 5008\mathring{A}$ line and the [Hβ] $\lambda = 4863\mathring{A}$ line. Source: SDSS-III/BOSS, DR12.

(b) Now that you understand the concept of redshift, can you relate redshift back to the scale factor that we discussed previously?

z	D (Mpc)	δD (Mpc)
0.014	49.356	5.221
0.018	83.433	8.059
0.02	68.444	6.296
0.026	103.59	11.43
0.026	113.59	10.45
0.03	112.54	10.35
0.036	135.31	13.0
0.043	172.71	14.30
0.045	209.57	18.31
0.05	211.51	17.51
0.05	217.44	20.00
0.052	202.92	16.80
0.063	280.11	23.19
0.071	291.97	26.86
0.075	305.73	28.12
0.079	284.01	23.51
0.088	452.21	37.44
0.101	427.89	45.27

Table 1: The three columns correspond to redshift (z) of the supernova and distance (D) of the supernova and the distance uncertainty (δD).

1.4. A First Measurement of H_0

To actually calculate the Hubble constant from observation, one has to determine both the redshift and the distance to far-away objects. (Can you explain why we cannot use nearby objects?) Astronomers use spectra to determine redshift just as we did in the last problem. However, the distance can only be obtained using indirect methods, such as standard candles. Given in Table 1 are the measured redshifts and distances to some of the nearest supernovae events. Use this set of data to calculate the Hubble constant and its uncertainty.

2. The Dynamics of the Universe

2.1 From Density to Expansion

Let us consider an observer in a uniform expanding medium, with mass-energy density ρ. Build a model to find a relationship between the Hubble constant and the mass-energy density of the universe. This is called the Friedmann equation and describes the expansion of the universe. This equation turns out to be the same when derived from the equations of general relativity as it is when derived from Newton's theory of gravity, so you can consider a classical approach.

2.2. An Introduction to Equations of State

Each form of energy in our universe evolves in a different way. For a perfect fluid we define the equation of state as,

$$w = p/\rho \tag{2}$$

where p is the pressure of the perfect fluid, ρ is its mass-energy density, and w is a nondimensional ratio.

Show that the density ρ of this fluid evolves with the scale factor a as,

$$\rho \sim a^{-3(1+w)} \tag{3}$$

2.3. Baryonic Matter and Photons

Now that you know how the mass-energy density evolves with the scale factor, in order to describe our universe, we need to determine the equations of state of each form of energy. Using thermal physics, show that the equations of state for normal matter is $w = 0$ while that for photons $w = 1/3$.

2.4. Exotic Forms of Energy

Now imagine dynamical fluid models in which we can obtain different values of w. Describe possible scenarios in which we could get $w > 1/3$ and $w < 0$ and focus on a model in which $w = -1$. This is in fact, the equation of state for dark energy and what you are describing is a model for the most mysterious form of energy in our universe.

2.5. An Accelerating Universe?

Using the equation of state w, derive an expression for the expansion acceleration of the universe as a function of density. Under what conditions does the expansion of the universe accelerate or decelerate?

2.6. Cosmology Dynamics

Since you now know the equations of state for different forms of energy, you can predict how universes in which some form of energy dominates over the others will evolve in time. Find the evolution of $\rho(t)$ and $a(t)$ assuming that the universe is dominated by some fluid with equation of state w. In particular, analyze the cases for which the universe is matter dominated($w = 0$), radiation dominated($w = 1/3$) and dark-energy dominated ($w = -1$). Also, for the specific case of radiation determine the evolution of the temperature of the universe with time, $T(t)$.

3. The Composition of the Universe

3.1. The Fate of the Universe

Before dark energy was discovered, there was a long debate over whether our universe will end up collapsing ("big crunch") or expanding forever("big freeze"). Describe what kind of universes can end up collapsing upon themselves and what kind of universes expand forever. Additionally, focus on the fate of universes that are matter-dominated, radiation-dominated and dark-energy dominated.

3.2. Changes in the Composition of the Universe

Our own universe has experienced a series of transitions, from radiation domination to matter domination to dark energy domination. Today the ratios between the three different forms of matter has been accurately measured: $0.73 \pm 0.04\%$ of the energy density of the universe takes the form of dark energy, $0.27 \pm 0.04\%$ is matter and $8.24 \times 10^{-5}\%$ is radiation. Assuming you can approximate these periods independently find and plot the evolution of the total energy density with respect to time, $\rho(t)$ and also determine and plot the evolution of the energy density for each individual component.

3.3. A Second Estimate for the Age of the Universe

Since the total energy of the universe is currently estimated to be $\rho = 9.47 \times 10^{-27} kg/m^3$ find a way to get the best estimate for the current age of the universe and determine the time scales for each period of domination.

3.4. Beyond Dark Energy?

Assume that our universe actually has other forms of matter which have not been so far detected but which might end up dominating the energy density of the universe in the far future. What equations of state would such form of matter have and can you find an appropriate physical model that would describe the behavior of such forms of matter?

4. Dark Matter

4.1. 80 year old results

Besides dark energy, our universe holds another great mystery, a form of matter called dark matter. Because dark matter has the same equation of state as normal, common baryonic matter, yet the two interact extremely rarely, the two forms of matter were very hard to distinguish. Today, we still do not know the exact nature of dark matter, although several candidates, ranging from supersymmetric particles to primordial black holes, have been proposed.

The first astrophysicist to postulate the existence of dark matter was Fritz Zwicky who analyzed the velocities and masses of stars in clusters. In 1930, Zwicky noticed that the average squared velocity of galaxies in the Coma nebulae is $\bar{v}^2 = 5 \times 10^{15} cm^2/s^2$, while by looking at the luminosity of the stars, he estimated that the mass of the cluster was $m = 5 \times 10^{10} M_\odot$ [1]. Explain why this data might imply that there is a missing form of matter in the cluster. Find a model to estimate the ratio between the amount of ordinary matter and dark matter in the cluster.

4.2. Galactic Rotation Curves

While Zwicky's method was controversial for many decades, a more definite demonstration of the existence of dark matter was the study of galaxy rotation curves by Vera Rubin in the early 1970s. A galaxy rotation curve shows the velocity of matter rotating in a spiral disk, as a function of radius from the center. Assuming that dark matter does not exist, create a simple theoretical model to illustrate the dependency of the rotation velocity of a galaxy with the distance from its center. Justify your assumptions. Note what happens at very large distances from the center of the galaxy.

4.3. Dark Matter Halo Models

The distribution of dark matter in and around galaxies has significant implications for galactic rotation curves and also for gravitational lensing by galaxies. It is thought that dark matter forms spherical halos around galaxies, as opposed to normal matter, which, as we see, forms flat disks in galaxies. One explanation for this spherical dark matter distribution is the very weak interaction between dark matter and normal matter, and also between dark matter particles themselves. Consider the following process of galactic formation: as dark matter and normal matter collect into roughly spherical clumps, the net angular momentum causes them to rotate. Because normal matter interacts electromagnetically with itself, the normal matter loses energy to friction and gradually collapses into a rotating disk. The dark matter, interacting only very weakly with itself and with the normal matter, remains in its original spherical shape. Create a model that estimate the density distribution of dark matter in this spherical halo. Use this model of dark matter distribution in galaxies to estimate the effect that dark matter has on rotation curves.

4.4. Measuring Rotation Curves

Table 2 presents the observational data of the low surface brightness galaxy $F583-1$, as noted by Blok, McGaugh, & Rubin (2001). The first column contains the distance from the center of the galaxy in kpc, the second one contains the corresponding rotation velocity in km/s and the third one gives the uncertainty in the measured velocity. Plot the rotation curve of the galaxy and compare it with the prediction of your theoretical model. How can you explain the difference between theory and observation? Estimate the amount of matter which is not visible.

5. Dark Matter — A Particle Or Not?

5.1. Detecting Dark Matter Particles

Scientists have long tried to detect the particle that could explain dark matter. Given a dark matter mass density in the Solar System ρ, a dark matter particle mass m, and a dark matter particle - nucleon interaction cross section σ, find an expression for the frequency of dark matter interactions with a single proton or neutron on Earth. Look up some rough estimates for ρ, m, and σ. Given these estimates, how frequently should a dark matter particle interact with 1 kg of argon on Earth? That is why so far, we have not managed to detect them.

5.2. An Alternative to Dark Matter

The modifified newtonian dynamics theory (MOND) attempts to explain the observed flat rotation curves without assuming the existence of dark matter. Instead, they change Newton's law of gravity. Build a model to find the approximate form of the acceleration in order to avoid the need for dark-matter. Describe the physical laws considered in your model and clearly state your assumptions. Besides particle detection,

[1] $M_\odot$ is the mass of our own Sun.

$R(kpc)$	$v_{rot}(km/s)$	$\sigma_v(km/s)$
0.1	1.1	11.1
0.4	10.0	7.0
0.8	17.4	9.6
1.1	23.5	11.2
1.4	31.0	5.2
2.0	40.9	5.9
2.3	44.8	4.5
2.8	51.7	12.7
3.1	55.7	4.5
3.7	62.0	4.5
4.3	66.6	6.9
5.3	72.3	15.5
5.5	73.4	6.0
6.3	77.1	12.1
7.2	80.3	13.5
9.3	84.7	6.0
14.0	86.9	5.6

Table 2: Radius and rotation velocity of galaxy $F583-1$ from Blok, McGaugh, & Rubin (2001).

describe a way in which one could realistically distinguish between MOND theories and the existence of dark matter.

6. On the Way to The Nobel Prize

As we have mentioned, in astronomy it is hard yet essential to measure distances to objects in the universe. One way of detecting distances is through the so-called "standard candles". These are objects whose luminosity we can predict, and by knowing both the luminosity $L(J/s)$ and the flux $f(J/s/m^2)$ of an object, we can calculate the distance to the object $d(m)$ by the relation

$$d = \sqrt{\frac{L}{4\pi f}} \tag{4}$$

Type Ia supernova are some of the best known "standard candles" in our universe. These events occur in binary systems where white dwarfs are in orbit around regular stars. The white dwarfs accrete mass from the regular stars by their gravitational pull. Once the mass of the white dwarfs reach $1.38M_\odot$, they collapse and explode forming supernovae. This behavior is thought to be standard throughout the universe, and its luminosity can be modeled. Using this method, astronomers have compiled tables of redshift and distances of observed Type Ia supernova as presented in the tables below. Here we use a quantity called the distance modulus μ, which relates to actual distance in Megaparsecs (Mpc) by,

$$\mu = 5\log_{10} D_L + 25 \tag{5}$$

Type Ia supernova are some of the most faraway objects with known distance and redshift.

6.1. Scatter Plots and Deviations.

To better visualize this data, make a scatter plot of all the Type Ia supernova from all three tables, Table 3—5, on a distance D_L versus redshift z plot, and then use a linear fit to calculate the Hubble constant and its error. Now, for each supernovae, subtract the distance predicted by the linear fit D_{fit} from the observed distance D_L to get the difference $\Delta D_L = D_{fit} - D_L$. Make a scatter plot of ΔD_L versus z for all the supernova events.

6.2. Something Interesting

Do you see something interesting? Is the scatter around zero random or is there some sort of trend? If there is a trend, explore its implication on the expansion of the universe.

Now, you have won the Noble prize...5 years ago!

Supernova Events	z	$\mu(\Delta\mu)$
1996E	0.43	41.74(0.28)
1996H	0.62	42.98(0.17)
1996I	0.57	42.76(0.19)
1996J	0.30	41.38(0.24)
1996K	0.38	41.63(0.20)
1996U	0.43	42.55(0.25)
1997ce	0.44	41.95(0.17)
1997cj	0.50	42.40(0.17)
1997ck	0.97	44.39(0.30)
1995K	0.48	42.45(0.17)

Table 3: High-z Supernova Ia Light Curve Parameters. The first column is the name of the supernova events. The second column denotes the redshift z of these events. The third column μ and $\Delta\mu$ denote the distance moduli of these events.

Supernova Events	z	$\mu(\Delta\mu)$
1995ao	0.30	40.74(0.60)
1995ap	0.23	40.33(0.46)
1996R	0.16	39.08(0.40)
1996T	0.24	40.68(0.43)
1997Ia	0.17	39.95(0.24)
1997apa	0.83	43.67(0.35)

Table 4: Medium-z Supernova Ia Light Curve Parameters. The first column is the name of the supernova events. The second column denotes the redshift z of these events. The third column μ and $\Delta\mu$ denote the distance modulus of these events.

Supernova Events	$\log_{10}$(cz)	$\mu(\Delta\mu)$
1994U	3.111	31.72(0.10)
1997bp	3.363	32.81(0.10)
1996V	3.870	35.35(0.17)
1994C	4.189	36.72(0.15)
1995M	4.202	37.12(0.15)
1995ae	4.308	37.58(0.21)
1994B	4.431	38.51(0.10)

Table 5: Nearby Supernova Ia Light Curve Parameters. The first column is the name of the supernova events. The second column gives the redshift in terms of $\log_{10}$(cz), where c is the speed of light $c = 3 \times 10^5 km/s$. The third column μ and $\Delta\mu$ denote the distance modulus of these events.