PRINCETON UNIVERSITY PHYSICS COMPETITION

Onsite Test 2016 Judging Directions

The test consists of **three** problems which took **1.5** hours to complete. **No collaboration** is allowed and partial credit will be given for incomplete solutions.

Some judging comments:

1. Partial credit is awarded and is important. Partial credit should be given consistently for the same problem on all tests.

2. Make your grading clear by circling it.

3. If the competitor clearly defines a symbol for the unknown answer in an early part of the problem and uses it in later parts, that is ok. Grade them on what they do correctly, not the final result being 100 percent correct.

Problem 1. A Jelly-Filled Universe *(10 points total)*

Suppose the Universe is filled with some strange gelatinous substance which produces a pseudo-drag force. A particle of mass m will experience a braking force (which we will also call the pseudo-drag force)

$$F = -ka, \tag{1}$$

where k is some positive constant and a is the particle's acceleration. For the following questions, assume the particle moves in one dimension for simplicity.

a) *(1 point)* Suppose there are no other forces on the particle. Describe the motion of the particle's position $x(t)$ for all times $t \geq 0$, assuming its initial speed $v(0) = v_0$ and initial position $x(0) = x_0$.

With no other external forces on the particle, $a = 0$. We can show this using Newton's second law:

$$-ka = ma,$$

which yields $a = 0$. Therefore,

$$x = x_0 + v_0 t.$$

b) *(1 point)* Now suppose the particle experiences a constant external force $F_{\text{ext}} > 0$ beginning at $t = 0$. Describe the motion of the particle's position $x(t)$ for all times $t \geq 0$, assuming its initial speed $v(0) = v_0$ and initial position $x(0) = x_0$.

Again, using Newton's laws we have

$$F - ka = ma,$$

which yields $a = F/(k + m)$. We see that the pseudo-drag force acts by contributing an effective mass to the particle. Integrating, we get

$$x = x_0 + v_0 t + \frac{1}{2} a t^2,$$

so

$$x = x_0 + v_0 t + \frac{1}{2}\left(\frac{F}{m + k}\right) t^2.$$

c) *(1 point)* What is the total work W done on the particle during the time interval $0 \leq t \leq T$ under this constant applied force? (Include the work done by the applied force and the pseudo-drag force.)

We know that the net force on the particle is constant. Defining the acceleration $a_0 = F/(k + m)$ from part b, we find

$$v(t) = v_0 + a_0 t$$

The net work done is therefore the change in kinetic energy during the time T:

$$W = \frac{1}{2} m v^2 - \frac{1}{2} m v_0^2 = \frac{1}{2} m a_0 T \left(a_0 T + 2 v_0\right)$$

d) *(2 points)* Describe the physical implication of the pseudo-drag force F for the particle. In particular, describe how this pseudo-drag force differs from kinetic friction (a constant braking force) and low-velocity viscous drag (a braking force proportional to the particle's speed). *Please be as specific as possible in your explanation.*

Kinetic friction is a constant braking force, so in a given time interval Δt the particle experiences a decrease in speed of $\Delta v \propto -\Delta t$. That is, the speed decreases at a constant rate. For viscous drag, the force increases linearly with speed, so $\Delta v \propto -v\Delta t$, which implies that the speed decreases exponentially fast $v \sim \exp(-kt)$. Unlike the other forces, the pseudo-drag force *only* acts in the presence of other forces. Alone, it has no effect on the system. In the presence of other forces, it acts by increasing the effective mass of the particle to $m \to m + k$.

e) *(1 point)* Suppose the particle is dropped near the Earth's surface where there is a gravitational field strength g. How long does it take (Δt) for the particle to fall through a height h? (Assume the force of gravity is mg downward, and does not depend appreciably on the height of the particle above the surface of the Earth.)

The force of gravity is $F_g = -mg$, where m is the particle's gravitational mass. Using Newton's second law, we find

$$a = -\left(\frac{m}{m+k}\right) g,$$

so the integrated equations give

$$\Delta h = -\frac{1}{2}\left(\frac{m}{m+k} g\right) \Delta t^2.$$

Solving for the time elapsed, we find

$$\Delta t = \sqrt{2h\left(\frac{m+k}{mg}\right)}.$$

f) *(2 points)* It is possible to recover the same result as in part e) for the movement of the particle by removing the pseudo-drag force F and replacing g with an effective gravitational field g_{eff}. Find this effective field for a given value of k and m. Consider the limits in which $m \to 0$ and $m \to \infty$. Do your answers make sense? Explain.

It is indeed possible to replace g with an effective gravitational field g_{eff}. Studying Newton's second law, we see that

$$a = -\left(\frac{m}{m+k}\right) g = -g_{\text{eff}},$$

which implies that we should set

$$g_{\text{eff}} = \left(\frac{m}{m+k}\right) g$$

Taking the limits, we get

$$\lim_{m\to 0} g_{\text{eff}} = 0,$$

$$\lim_{m\to \infty} g_{\text{eff}} = g.$$

These results make sense. As $m \to 0$, the particle becomes massless, so it is no longer affected by gravity. So, it doesn't matter whether there is a gravitational field ($g \neq 0$) or not ($g_{\text{eff}} = 0$). In the second case, we have that the particle is infinitely massive. We no longer expect the the pseudo-force, which contributes a constant, finite term k to the effective mass, to matter. Thus, we expect $g_{\text{eff}} = g$, since the pseudo-drag force no longer contributes.

g) *(2 points)* In electromagnetism, the Abraham-Lorentz force is a braking force that depends on the *derivative* of the particle's acceleration:

$$F_{AL} = mq\frac{da}{dt}, \tag{2}$$

where q is some constant and m is the particle's mass, included for convenience. Including a time-varying external force $F_{\text{ext}}(t)$, the equation of motion is

$$F_{\text{ext}}(t) + mq\frac{da(t)}{dt} = ma(t). \tag{3}$$

We can integrate this expression to find the solution for $a(t)$,

$$a(t) = \frac{1}{mq}\int_t^\infty e^{-\frac{t'-t}{q}}F_{\text{ext}}(t')dt'. \tag{4}$$

Suppose a constant external force $F_{\text{ext}}(t) = F_{\text{ext}}$ is turned on at some faraway time $t = T > 0$ and lasts for all $t \geq T$. What is the acceleration $a(t)$ of the particle for $t \geq 0$ according to the solution for $a(t)$ given above? What is strange about this situation? *Please be specific by commenting on any times of interest.*

Plugging in F_{ext} yields the acceleration $a(t)$ for $T > t \geq 0$:

$$a(t) = \frac{F_{\text{ext}}}{m}e^{-\frac{T-t}{q}}$$

This is quite a strange result. First, notice that as $T \to \infty$, we find that $a(t) \to 0$. This is expected, since a force applied in the infinitely far future should not affect the acceleration at any finite time t. Now consider the force for $t - T \equiv s \geq 0$. Now, we get

$$a(t) = \frac{F_{\text{ext}}}{m}$$

an acceleration which is constant in time (this solution comes from the limits of the integral no longer being T and ∞ but rather t and ∞). Now let's consider the weird result. Notice that for $T - t \equiv r \geq 0$, we get

$$a(t) = \frac{F_{\text{ext}}}{m}e^{-\frac{r(t)}{q}}.$$

This means that the acceleration $a(t)$ for $t < T$ is finite and exponentially small, but still finite! So, a force in the future can influence the particle's acceleration (and hence its motion) in the past! Obviously this is noncausal and nonphysical. The Abraham-Lorentz force signifies a breakdown of classical mechanics in conjunction with electromagnetism and hints as the necessity of quantum physics.

Problem 2. Measuring an Unknown Force Using Friction *(8 points total)*

Consider a block of mass m moving on a horizontal flat surface with friction under the influence of gravity. The block is confined to only move in the x direction. The surface has a coefficient of kinetic friction, μ, which is equal to its coefficient of static friction. Imagine a force $f(t)$ is applied to the block in the $+x$ direction starting at $t = 0$ and ending at $t = T$. Note: $f(t)$ is restricted to be positive and to the right, but may be 0. Moreover, $f(t)$ is monotonically increasing.

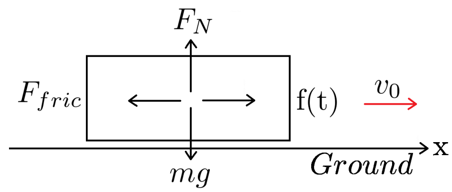

Figure 1: Diagram of block on a surface with friction.

You next perform a few experiments. You start the block with some initial velocity $v_0 \geq 0$ moving to the right at $t = 0$. Next, $f(t)$ is applied and, after waiting for some time until $t = T$, you observe v_f, the final velocity. You repeat the experiment twice to try to determine the form of $f(t)$ with only these two measurements.

a) *(1 point)* Before we discuss the results of the experiments, describe in words all forms of $f(t)$ that you can think of that produce meaningfully distinct forms of motion for the block. Meaningfully distinct is up to your own interpretation, but think about what the block does for a few example forces in limiting cases, and use this information to inform your interpretation of meaningfully distinct. (You may also find it helpful to do later parts of this problem and come back to part a) if you are stuck.)

Answer is up to the judge's discretion. Ideally they should briefly describe that there are 3 qualitatively different outcomes: Block can never stop regardless of initial speed. Block can stop for some initial velocities that are small enough. In the case when the block does stop, there are 2 outcomes possible: it can accelerate for some time at the end or it might never accelerate again if it stops. See Fig. 2.

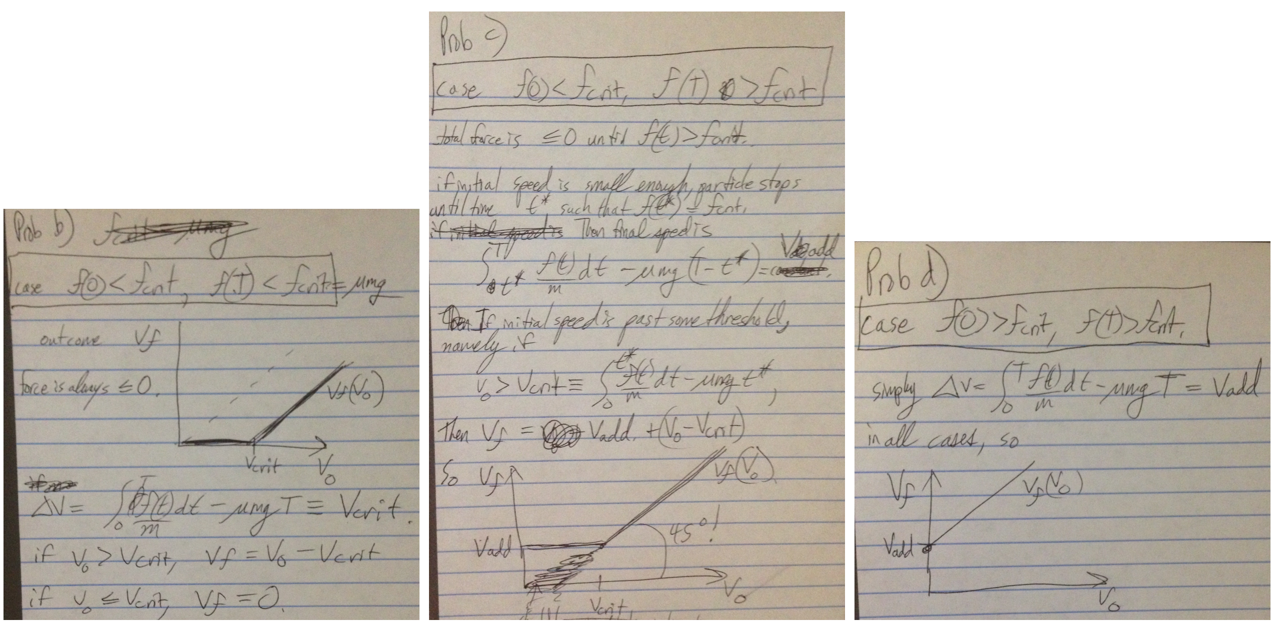

Figure 2: Diagrams of possible conceptual cases.

b) *(1 point)* Now we consider a specific form of $f(t)$ which will have the designation $f_1(t)$. This is just some specific function, although we will not tell you what it is yet. After performing the experiment twice, you obtain the following results:

$$\text{When } v_0 = 0 \text{ m/s}, \; v_f = 0 \text{ m/s}.$$
$$\text{When } v_0 = 3 \text{ m/s}, \; v_f = 2 \text{ m/s}.$$

By considering all possible cases of the form of $f_1(t)$ that are consistent with the result you obtained above after your experiment, write down a possible graph of v_f with respect to v_0 (v_0 on the x axis and v_f on the y axis). This graph describes the resulting v_f you would observe if you initialized the box with speed v_0 at time $t = 0$, allowed the box to evolve under the force $f_1(t)$, and then observed the final speed v_f at time T. Recall that $v_0 \geq 0$ by construction in this problem.

Answer: the important points are showing exact graph in terms of correct x and y intercept and location of kink. Also need asymptotic angle of graph to be 45 degrees.

c) *(1 point)* Someone now changes the form of $f(t)$ to be something else, call it $f_2(t)$. You perform the same experiment again, this time obtaining different results. After performing the experiment twice, you obtain the following results:

$$\text{When } v_0 = 0 \text{ m/s}, \; v_f = 1 \text{ m/s}.$$
$$\text{When } v_0 = 3 \text{ m/s}, \; v_f = 2 \text{ m/s}.$$

By considering all possible cases of the form of $f_2(t)$ that are consistent with the result you obtained above after your experiment, write down a possible graph of v_f with respect to v_0 (v_0 on the x axis and v_f on the y axis). Recall that $v_0 > 0$ by construction in this problem.

Answer: the important points are showing exact graph in terms of correct x and y intercept and location of kink. Also need asymptotic angle of graph to be 45 degrees.

d) *(1 point)* Someone now changes the form of $f(t)$ to be something else, call it $f_3(t)$. You perform the same experiment again, this time obtaining different results. After performing the experiment twice, you obtain the following results:

$$\text{When } v_0 = 0 \text{ m/s}, \; v_f = 1 \text{ m/s}.$$
$$\text{When } v_0 = 3 \text{ m/s}, \; v_f = 4 \text{ m/s}.$$

By considering all possible cases of the form of $f_3(t)$ that are consistent with the result you obtained above after your experiment, write down a possible graph of v_f with respect to v_0 (v_0 on the x axis and v_f on the y axis). Recall that $v_0 > 0$ by construction in this problem.

Answer: the important points are showing exact graph in terms of correct x and y intercept and location of kink. Also need asymptotic angle of graph to be 45 degrees.

The answers to b), c), and d) are shown in the below Fig. 3.

e) *(1 point)* Are the graphs you drew for parts b), c), and d) the unique possible graphs consistent with the experimental data? Explain why or why not.

Answer: yes, they are unique and the students needs to say more than just literally the word yes, but not much more.

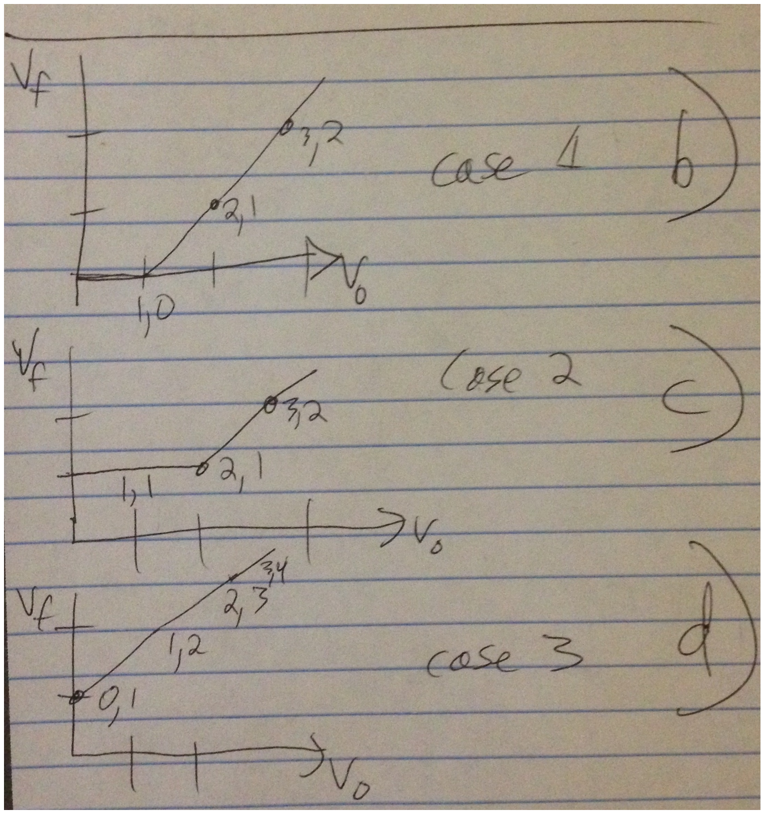

Figure 3: Diagram of graphs of v_f vs. v_0 for the cases considered.

f) *(2 points)* Write down 3 possible graphs of $f_1(t)$, $f_2(t)$, and $f_3(t)$ with $f(t)$ on the y axis and t on the x axis. For each axis, you should label one point (for example the place where $t = T$) using a combination of parameters given in the problem. This one point determines the scale of forces and time on which you draw the function $f(t)$. You should draw each $f(t)$ as a monotonically increasing line for simplicity in grading, although in principle $f(t)$ could be some other function.

Answer: they need to label $\mu\ g$ on f axis and T on time axis. Need to have function $f(t)$ be monotonically increasing. Need to have them pass threshold of $f(t) = \mu g$ or not depending on which force the are considering. See Fig. 4. This problem is somewhat confusing/hard because we have not told them what

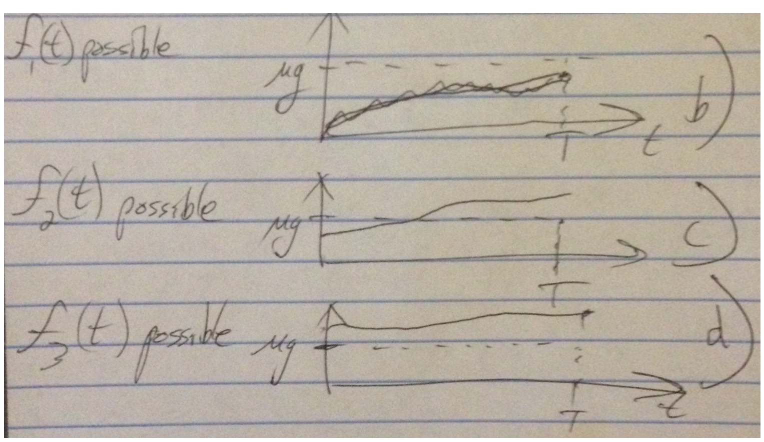

Figure 4: Diagram of forces in all 3 cases.

μ, g, m, and T are specifically, so they do not have all of the information to determine the form of $f(t)$ explicitly. However, only certain function $f(t)$ are possible. They must stay on one or the other side of the threshold of μg.

g) *(1 point)* What is the optimal set of two v_0 to measure in order to precisely determine the graph of $v_f(v_0)$? Describe why.

Answer is $v_0 = 0$ and $v_0 =$ infinity or very large. As long as you measure a v_0 value large enough to be past the kink in the graph, you uniquely determine the graph's shape.

Problem 3. Magnetic Monopoles and Charge (Hard) *(12 points total)*

Consider a cylindrical coordinate system with coordinates z, θ, and ϕ fixed to the end of a very long solenoid as shown in the figure below. The origin of the coordinate system is on the axis of the cylinder and at the end of the solenoid. The solenoid's diameter is D, a current of intensity I traverses the solenoid's coils, and the number of coils per unit length is n.

At an *external* point near the top end of the solenoid, having position vector $\vec{r}$ (where the modulus, or magnitude, of the position vector $|\vec{r}| = r$ is much smaller than the length of the solenoid but $r >> D$), the magnetic field vector can be expressed, approximately, as

$$\vec{B}(\vec{r}) \approx \lambda \frac{1}{r^2} \hat{e_r} \tag{5}$$

(The notation $\hat{e_r}$ means a vector of unit length pointing in the direction of $\vec{r}$ from the origin of the coordinate system, which is defined in the figure below.) This magnetic field is similar to the magnetic field generated by a magnetic monopole, as is described below.

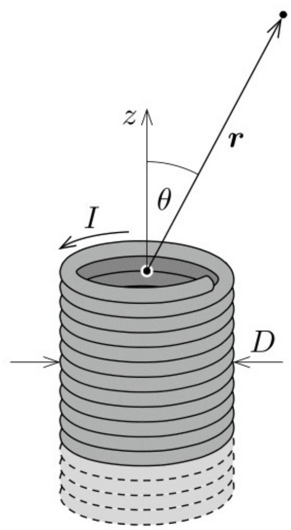

Figure 5: An infinitely long solenoid and a cylindrical coordinate system.

For your reference, we will now provide a few formulas from electricity and magnetism which may be useful in completing this problem:

$$\vec{F} = q(\vec{v} \times \vec{B}) \quad - \quad \text{Force on a moving charge with velocity } \vec{v} \text{ in a magnetic field } \vec{B} \tag{6}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{e_r} \quad - \quad \text{Electric field generated by a charge } q, \text{ an electric monopole} \tag{7}$$

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{m_0}{r^2} \hat{e_r} \quad - \quad \text{Magnetic field that would be generated by a magnetic monopole } m_0 \tag{8}$$

$$\vec{p_e} = q\vec{d} \quad - \quad \text{Electric dipole caused by charges } q \text{ and } (-q) \text{ that are separated by } \vec{d} \tag{9}$$

$$\vec{p_m} = I \cdot (\vec{A}) \quad - \quad \text{Magnetic dipole of a loop of current } I \text{ with area vector } \vec{A} \tag{10}$$

You may also find it useful to know that an electric dipole $\vec{p} = q\vec{d}$ can be thought of as two electric monopoles (charged particles) of opposite charge that are separated by a distance $\vec{d}$. Hypothetically it would be possible to construct an analogous magnetic dipole from two magnetic monopoles as $\vec{p_m} = m_0\vec{d}$, which may be useful in answering part a). One further note: the area vector of a loop of current is in the direction perpendicular to the loop. For example, it is in the positive z direction in the figure above for a loop in the solenoid, and has a magnitude which is the area of the loop.

a) *(2 points)* Determine the value of λ in the equation for the magnetic field as a function of D, I, n, and universal constants.

Answer: Far from the solenoid each turn acts as a magnetic dipole m. This can be imagined as two opposite magnetic monopoles $\pm P$ at a distance $d = \frac{1}{n}$, which is the distance between two adjacent turns. Considering the whole solenoid we obtain a chain of such dipoles where the opposite poles of the neighbouring dipoles exactly cancel each other, except at the ends. The magnetic dipole of a turn is $m = \frac{\pi}{4}D^2 I$ so the strength of the magnetic monopole at the end of the solenoid is $P = mn = \frac{\pi}{4}D^2 In$. Using an analogy between electrostatics and magnetostatics, $\frac{1}{\epsilon_0}$ becomes μ_0. Thus the magnetic field is:

$$\vec{B}(\vec{r}) = \frac{\mu_0 P}{4\pi}\frac{\hat{e_r}}{r^2}$$

Hence $\lambda = \frac{\mu_0 D^2 In}{16}$.

Consider a particle of mass m and charge Q moving around the top end of the solenoid. We will assume that the equation for the strength of the magnetic field, Eq. (5), holds true. Neglect the effects of gravity in this problem. Also note that there are no electric field forces on the particle because the solenoid only produces magnetic field. The angular momentum $\vec{L} = m\vec{r} \times \vec{v}$ of the charged particle is not conserved during its motion, however the vector $\vec{J}$ defined by

$$\vec{J} = \vec{L} + C\hat{e_r} \tag{11}$$

is conserved (C is a constant).

b) *(2 points)* Find C as a function of Q, λ, and universal constants. Hint: The following formula might be useful: $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$

Answer: $\vec{J}$ is constant, thus $\frac{d\vec{J}}{dt} = 0$

$$\frac{d\vec{J}}{dt} = m\frac{d(\vec{r} \times \vec{v})}{dt} + C\frac{d}{dt}(\frac{\vec{r}}{r}) = m(\vec{v} \times \vec{v} + \vec{r} \times \vec{a}) + C\frac{1}{r^2}(r\vec{v} - \frac{dr}{dt}\vec{r}) =$$

$$= \frac{\lambda Q}{r^3}(\vec{r} \times (\vec{v} \times \vec{r})) + \frac{C}{r^3}(r^2\vec{v} - (\vec{r} \cdot \vec{v}) \cdot \vec{r}) = \frac{r^2\vec{v} - (\vec{r} \cdot \vec{v}) \cdot \vec{r}}{r^3}(\lambda Q + C)$$

Thus $C = -\lambda Q$.

The vector $\vec{J}$ can be viewed at the sum of the angular momentum of the particle and the electromagnetic field. Thus, $\vec{J}$ is the total angular momentum of the system formed by the particle and the electromagnetic field. According to Quantum Mechanics, the projection of this total angular momentum vector on an arbitrary direction (for example the direction of $\hat{e_r}$) is quantized as a multiple of $\hbar/2$ ($\hbar$ is simply a number from Quantum Mechanics, the reduced Planck constant). For a fixed magnetic monopole strength m_0, the quantization condition $\vec{J} \cdot \hat{e_r} = \hbar/2$ is a constraint on the relationship between the particle charge Q and m_0.

c) *(2 points)* Find the relationship between the strength of an ideal magnetic monopole, m_0, which produces the same magnetic field as in Eq. (5) and the charge Q of a particle orbiting that ideal monopole. Express this result as a function of universal constants. Note that, if such an ideal magnetic monopole existed (not just the approximate magnetic monopole field produced by the long solenoid), this relationship implies that the charge of the particle must be quantized. Indeed, in 1931, Paul Dirac similarly showed that if there were at least one magnetic monopole in the universe, this would explain the quantization of electric charge. (For this problem, if you did not answer part b) or part a), so that you do not know λ or

C, you may assume $C = \gamma Q m_0$ for a proportionality constant γ, the charge Q, and magnetic monopole strength m_0. Leave the value of the constant γ in your answer.)

Answer: $\vec{J} \cdot \hat{e_r} = -\lambda Q = n\frac{\hbar}{2}$ where n is an integer. Denoting the elementary charge by e and using $\lambda = \frac{\mu_0}{4\pi} P$, we get:
$\frac{\mu_0}{4\pi} P_0 e = \frac{\hbar}{2}$, which yields $P_0 = \frac{h}{\mu_0 e} = 3.292 \cdot 10^{-9}$ Am.

We will now return to exploring the motion of the charged particle in the magnetic field of Eq. (5). It can be shown that the particle moves on a path that lies on a conical surface. The vertex of the cone is the origin of the coordinate system we defined above.

d) *(2 points)* Find the angle β of the cone (the angle of the opening of the tip of the cone) as a function of the magnitude of the initial angular momentum, L_0, as well as Q and λ. (Again, if you have not found C above, you may treat it as given for this part of the problem and express your answer in terms of C.)

Answer: $\vec{r} \times \vec{v} \perp \hat{e_r}$ thus $\vec{J} \cdot \hat{e_r} = C$
$\vec{J}$ is conserved, thus $|\vec{J}| = \sqrt{L_0^2 + C^2}$
Hence $\cos\beta = \frac{|\vec{J} \cdot \hat{e_r}|}{|\vec{J}|} = \frac{\lambda Q}{\sqrt{L_0^2 + \lambda^2 Q^2}}$

Assume the particle initially is on the z axis, at height r_0, and has initial velocity $\vec{v_0}$ which is oriented at an angle $\alpha_0 < \frac{\pi}{2}$ with respect to the negative z axis. (Therefore, its initial velocity component in the negative z direction is $v_0 \cos(\alpha_0)$.)

e) *(2 points)* Find the minimum distance r_{min} between the particle and the origin of the coordinate system as a function of r_0 and α_0.

Answer: $\vec{F} \perp \vec{v}$ thus $|\vec{v}| = v_0$ is constant.
$\vec{J}$ is conserved, thus $m^2 r^2 v_0^2 \sin\alpha^2 + C^2 = m^2 r_{min}^2 v_0^2 + C^2$, where $\sin\alpha_{min} = 1$.
Hence $r_{min} = r_0 \sin\alpha_0$.

f) *(2 points)* Find the time τ it takes the particle to go from r_0 to r_{min} as a function of r_0, α_0, and v_0.

Answer: $\frac{dr}{dt} = -v_0 \cos\alpha$ and $r \sin\alpha = r_0 \sin\alpha_0$
$\frac{dr}{dt} = -v_0 \sqrt{1 - \frac{r_0^2 \sin{\alpha_0}^2}{r^2}} \Rightarrow \frac{dr}{\sqrt{1 - \frac{r_0^2 \sin{\alpha_0}^2}{r^2}}} = -v_0 dt \Rightarrow r_0 \cos\alpha_0 = v_0 \tau$
Thus $\tau = \frac{r_0 \cos\alpha_0}{v_0}$.
The initial velocity points toward the origin, thus the value of τ that results is positive.